Quiz 8 Solutions — 26 October 2018 Name:

Question 1 (2 pts)**.** Mark each statement true or false, and give a brief justification of your answer.

(a) If an $n \times n$ matrix A is invertible, then its inverse can be written as a linear combination of $I, A, A^2, \ldots, A^{n-1}$.

True. This follows from the Cayley-Hamilton theorem. In particular it was shown on the homework that if A is invertible with characteristic polynomial $(-1)^n t^n + a_{n-1}t^{n-1} + \cdots + a_0$ then $a_0 \neq 0$ and $A^{-1} = -\frac{1}{a_0}((-1)^n A^{n-1} + a_{n-1}A^{n-2} + \cdots + a_1 I)$.

(b) Let T be a linear operator on a vector space V. If every non-trivial T-invariant subspace of V contains an eigenvector, then T is diagonalizable.

False. It was proven on the homework that this is true whenever the characteristic polynomial of T splits, and for this T need not be diagonalizable. Concretely, the linear map on $\mathbb{R}^2$ given by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is a counterexample.

Question 2 (4pts)**.** Consider the matrix

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix},$$

and T the linear operator on $V = M_{2\times 2}(\mathbb{R})$ given by left multiplication by A, i.e. $T(M) = AM$. Find a basis for the T-cyclic subspace generated by the identity matrix I.

The T-cyclic subspace generated by I is

$$\text{Span}(I, T(I), T^2(I), \ldots) = \text{Span}(I, A, A^2, \ldots)$$

because $T(I) = AI = A$, and $T^2(I) = A(AI) = A^2$, etc. A basis is given by $\{I, A\}$. It is possible to see this by computing A^2 and showing that it is a linear combination of A and I; a faster way is to use the Cayley-Hamilton theorem, which says that $\text{ch}_A(A) = 0$, and since ch_A is a degree 2 polynomial, the equation $\text{ch}_A(A) = 0$ shows that I, A, A^2 are linearly dependent. More concretely, the characteristic polynomial of A is $(-t+1)(-t+4) - 6 = t^2 - 5t - 2$, and thus

$$A^2 = 5A + 2I$$

gives A^2 as a linear combination of A and I.

Question 3 (4 pts)**.** Let T be a diagonalizable linear operator on a finite-dimensional inner product space V. Show that if $\langle Tx, x\rangle = 0$ for all $x \in V$, then $T = 0$.

Since T is diagonalizable, we can choose a basis $\beta = \{v_1, \ldots, v_n\}$ of V consisting of eigenvectors for T, say $Tv_i = \lambda_i v_i$. Combining this equation with the fact that $\langle Tx, x\rangle = 0$ for any $x \in V$, and using the linearity of the inner product in the first slot, we find

$$0 = \langle Tv_i, v_i\rangle = \langle \lambda_i v_i, v_i\rangle = \lambda_i \langle v_i, v_i\rangle.$$

One of the axioms of an inner product is that $\langle x, x\rangle = 0$ iff $x = 0$; since $v_i \neq 0$ it must be that $\lambda_i = 0$. Since this is true for all λ_i, we see that $Tv_i = 0$ for $v_1, \ldots, v_n$, and this implies $T = 0$.